16.8 Stokes' Theorem

Stokes' Theorem

A **second higher-dimension** analog of Green's Theorem is called **Stokes's Theorem**, after the English mathematical physicist George Gabriel Stokes.

Stokes' Theorem gives the **relationship between** a **surface integral** over an oriented surface *S* **and a line integral** along a closed space curve *C* forming the boundary of *S*.

The **positive direction** along *C* is **counterclockwise** relative to the **normal vector N**. Grasping the **normal vector N** with your right hand, your thumb will point in the **direction of N**, your fingers will point in the **positive direction C**.

Sketch:

Stokes' Theorem

Another way to look at it is if you **walk in the positive direction around *C*** with your **head** pointing in the **direction of N**, rather the unit normal **n**, then the surface will **always be on your left**.

Stokes' Theorem

Let *S* be an **oriented surface** with unit normal vector **N**, bounded by a piecewise smooth simple **closed curve *C*** with a **positive orientation**. If **F** is a vector field whose component functions have continuous first partial derivatives on an open region containing *S* and *C*, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

LHS is the **line integral of the boundary**, while the

RHS is the **surface integral of the curl**.

Stokes' Theorem

Also, we note that since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \quad \text{and} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

Stokes' Theorem says that the line integral around the boundary curve of *S* of the **tangential component of F** is equal to the surface integral over *S* of the **normal component of the curl of F**.

The **positively oriented** boundary curve of the oriented surface *S* is often written as ∂S, so Stokes' Theorem can be expressed as

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Example – Using Stokes' Theorem

Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

$\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + x\,\mathbf{j} + e^{xy}\cos z\,\mathbf{k}$,

S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geqslant 0$, oriented upward

Example – Using Stokes' Theorem

Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

$\mathbf{F}(x, y, z) = yz^3\,\mathbf{i} + \sin(xyz)\,\mathbf{j} + x^3\,\mathbf{k}$,

S is the part of the paraboloid $y = 1 - x^2 - z^2$ that lies to the right of the xz-plane, oriented toward the xz-plane

Stokes' Theorem

Generally, if S_1 and S_2 are **oriented surfaces** with the **same oriented boundary** curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Useful when **difficult to integrate** over one surface **but easy to integrate** over the other.

Example – Using Stokes' Theorem

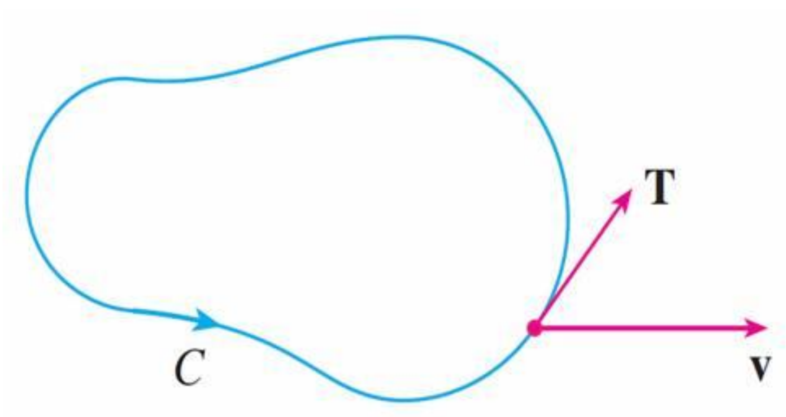
Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + e^z\,\mathbf{j} + xy^2\,\mathbf{k}$,

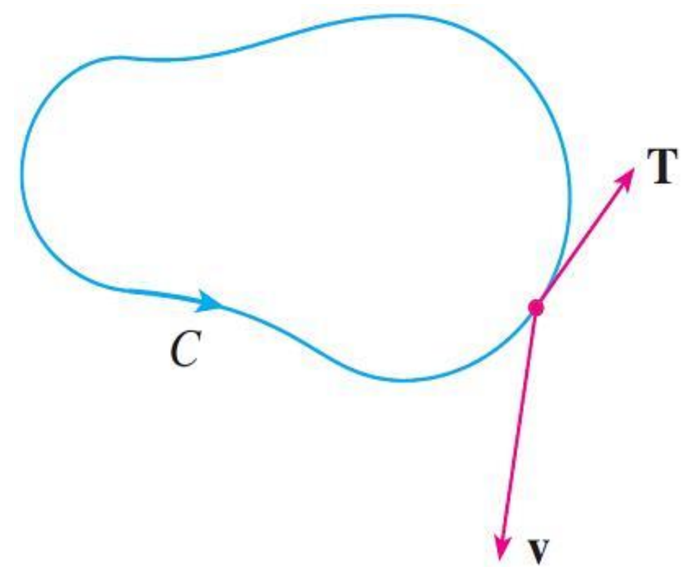
S consists of the four sides of the pyramid with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$ that lie to the right of the xz-plane, oriented in the direction of the positive y-axis

Stokes' Theorem (Book)

$\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of $\mathbf{v}$ around C.



(a) $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$, positive circulation



(b) $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$, negative circulation

Stokes' Theorem

We know that **F** is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path *C*. Given *C*, suppose we can find an orientable surface *S* whose boundary is *C*.

Then Stokes' Theorem gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0.

Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve *C*.

Example – Using Stokes' Theorem

Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Note: C is oriented counterclockwise as viewed from above.

$\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$,

C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$

Example – Using Stokes' Theorem

Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Note: C is oriented counterclockwise as viewed from above.

$\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + 4x\,\mathbf{j} + 5y\,\mathbf{k}$,

C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$

Example – Using Stokes' Theorem

Verify that Stokes' Theorem is true for the given vector field **F** and surface *S*.

$\mathbf{F}(x, y, z) = 3y\,\mathbf{i} + 4z\,\mathbf{j} - 6x\,\mathbf{k}$,

S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy-plane, oriented upward